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THE EFFECT OF RADIATION PRESSURE ON THE SECULAR ACCELERATION OF SATELLITES

by

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Summary. --Satellite accelerations play a crucial role in determining the structure of the high atmosphere, and it is therefore important to assess and eliminate the effect of perturbing forces that compete with air drag and that may therefore confuse our picture of the thermosphere. In particular, this study evaluates the effect of solar radiation pressure on the secular acceleration of earth satellites. For perigee heights less than about 800 km the period changes due to radiation pressure are minor compared with those due to atmospheric drag. At greater heights and lower air densities, radiation pressure becomes increasingly important. When a satellite is in sunshine all around its orbit, the period change arising from the pressure of sunlight is zero. But during the weeks or months it is penetrating the earth's shadow and is therefore exposed to a photon wind only part of each circuit, the secular acceleration may attain substantial values, positive or negative depending on the orientation of the orbit relative to the sun. Several special cases of orientation are discussed, and a general formula for computing secular accelerations due to radiation pressure is derived as far as terms in the square of the eccentricity.

1. Introduction

The secular perturbations of a satellite orbit arising from solar radiation pressure have been discussed recently (Musen, 1960; Musen, Bryant, and Bailie, 1960; Parkinson, Jones, and Shapiro, 1960). In particular, the variations of 1 or 2 km in the perigee height of Satellite 1958 $\beta 2$ (Vanguard I) predicted by Musen and his collaborators, when combined with the gravitational effects of the sun and moon, agree very well with the observed changes during the first two years in orbit. Very recently, much larger variations of eccentricity and perigee height of Satellite 1960 $\iota 1$ (Echo I) during its early life have been observed and found to be in excellent accord with theory (Jastrow and Bryant, 1960; Shapiro and Jone, 1960).

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The problem to be considered here concerns the short-term secular variations in period to be expected from solar radiation pressure. The secular acceleration of a satellite as a function of the time can often be derived from observation with considerable accuracy. As is well known, these accelerations, $\Delta P/P$, may then be employed to deduce values of $\rho_q\sqrt{H_q}$, where ρ_q and H_q are the density and scale height of the atmosphere at the locations of perigee. The question naturally arises how high in the atmosphere it is legitimate to deduce these parameters from the observed period changes. Are there other perturbing forces, in addition to atmospheric drag, which will produce finite values of $\Delta P/P$? If so, how can their effects be eliminated in order to avoid erroneous conclusions about the structure of the thermosphere? In what follows it will be shown that the effect of solar radiation on the week-to-week variation in period is negligible when a satellite is continually in sunshine. Later, however, because of the motion of sun, node, and perigee, the satellite must spend some of its time passing through the earth's shadow; the secular acceleration due to the force of sunlight may then exceed that due to atmospheric drag at heights above 800 km or so.

2. The Perturbing Acceleration due to Radiation Pressure

A satellite of average physical cross-section A and mass m at distance $r_\odot = 1$ a.u. from the sun intercepts energy at the rate $L_\odot A/4\pi r_\odot^2$, where $L_\odot$ is the total power output of the sun. Hence the momentum gained per unit time, or repulsive force, is of amount $L_\odot A/4\pi c r_\odot^2$, where c is the velocity of light. If the incident energy is reflected specularly or is absorbed and re-emitted isotropically, virtually no net momentum is carried away. We therefore assume that the radial acceleration has a magnitude

$$f = (A/m) L_\odot / 4\pi c r_\odot^2 . \qquad (1)$$

We also assume for simplicity that during the few hours a satellite spends revolving once around the earth the vector $\vec{f}$ is a constant relative to the satellite's orbit. We thus ignore a variety of small effects: (1) possible variations of the solar constant, (2) the minuscule change in the solar distance, (3) the slight motion of the sun in right ascension and declination, (4) the motion of the satellite's node, (5) the motion of the satellite's perigee, and (6) the Poynting-Robertson drag. The magnitude of $\vec{f}$ is probably constant to good accuracy. The change in direction of $\vec{f}$, through items (3), (4), and (5), amounts at most to a few tenths of a degree during one typical orbital period. Item (6) modifies the direction of $\vec{f}$ by less than a minute of arc. The influence of all these factors on the secular acceleration of a satellite is ordinarily much smaller than that due to the earth's shadow, and we shall not consider them here.

A more important effect appears to be re-radiation of sunlight from the earth. When the sun is overhead, a satellite at a typical height experiences an upward push due to the reflected component of sunlight amounting to at least 20 percent of the downward push of direct sunlight. When the sun is at larger zenith distances, the effect is less important, but is complicated by the fact that the repulsive force is no longer quite radial from the center of the earth. Over the entire earth the magnitude of the mean outward acceleration due both to the reflected sunlight and to the infra-red radiation by the surface and atmosphere is less than 20 percent of that due to direct sunlight for satellites with perigee heights greater than 800 km. Although it is desirable that the influence of terrestrial re-radiation be calculated, we shall not attempt to do so in this study.

3. The Secular Acceleration of a Satellite

The instantaneous time rate of change of semi-major axis of an earth satellite is given (e.g. Moulton, 1914; Smart, 1953) by

$$\frac{da}{dt} = \frac{Pe \sin\theta}{\pi\sqrt{1 - e^2}} R + \frac{P(1 + e\cos\theta)}{\pi\sqrt{1 - e^2}} S , \tag{2}$$

where P is the orbital period, e the eccentricity, θ the true anomaly, R the component of $\vec{f}$ directed radially away from the center of the earth, and S the component in the satellite's orbit plane at right angles to the radius vector and making an angle less than 90° with the velocity vector of the satellite. To change this expression to the rate of change of period with true anomaly, we make use of the law of areas, the polar equation of the orbit, and the derivative of Kepler's third law. Substitution gives

$$\frac{dP}{d\theta} = \frac{dP}{da}\frac{da}{dt}\frac{dt}{d\theta} = \frac{3Pa^2(1 - e^2)}{GM_{\oplus}}\left[\frac{Re\sin\theta + S(1 + e\cos\theta)}{(1 + e\cos\theta)^2}\right]. \tag{3}$$

The components of the disturbing acceleration may be deduced from Figures 1 and 2. The xy-plane coincides with the orbit plane of the satellite, Q is the direction of perigee, and P the instantaneous position of the satellite. The direction of the sun is S, inclined by an angle $i' \equiv \widehat{ZS}$ from the orbit normal. The direction J defines the direction of the x-axis and is the intersection of the orbit plane and a perpendicular plane which contains the sun. The instantaneous true anomaly of the satellite is $\theta \equiv \widehat{QP}$; we define the angle $\beta \equiv \widehat{JQ}$. Figure 2 shows that the total magnitude of the perturbing acceleration in the orbit plane is f sin i'; the radial and transverse components are therefore

$$\begin{aligned} R &= -f \sin i' \cos(\theta + \beta) , \\ S &= +f \sin i' \sin(\theta + \beta) . \end{aligned} \tag{4}$$

Substituting equations (4) in equation (3) and integrating around the orbit, we find that the secular acceleration of an earth satellite due to the pressure of sunlight is

$$\frac{\Delta P}{P} = \frac{1}{P}\int_0^{2\pi} \frac{dP}{d\theta}\, d\theta = \frac{3a^2(1 - e^2) f \sin i'}{GM_{\oplus}} \int_{0,\ \theta_2}^{\theta_1,\ 2\pi} \frac{[\cos\beta\sin\theta + \sin\beta(e + \cos\theta)]}{(1 + e\cos\theta)^2}\, d\theta . \tag{5}$$

Again, it should be stressed that the assumptions implicit in this formulation are that $\vec{f}$ has a constant magnitude as the orbit is described and also that the angles i' and β are constant during this interval. The limits of integration on the right side of equation (5) account for the fact that in the general case a satellite will enter the earth's shadow when the true anomaly is θ_1 and emerge when it is θ_2.

During this time, of course, the perturbing acceleration vanishes. The first term in the integrand is readily evaluated, while the second term is found on substituting the well-known relation

$$\tan(\theta/2) = (1 + e)^{1/2}(1 - e)^{-1/2}\tan(E/2),$$

where E is the eccentric anomaly. The value of the integral is

$$\left.\frac{e^{-1}\cos\beta + \sin\beta\sin\theta}{1 + e\cos\theta}\right|_{0,\ \theta_2}^{\theta_1,\ 2\pi} = -\left.\frac{e^{-1}\cos\beta + \sin\beta\sin\theta}{1 + e\cos\theta}\right|_{\theta_1}^{\theta_2}$$

$$= \left[-\frac{\cos\beta}{e} + \frac{\cos(\beta + \theta)}{1 + e\cos\theta}\right]\Bigg|_{\theta_1}^{\theta_2} = \left.\frac{\cos(\beta + \theta)}{1 + e\cos\theta}\right|_{\theta_1}^{\theta_2}. \tag{6}$$

When equations (1) and (6) are substituted in equation (5), the expression for the secular acceleration of a satellite due to solar radiation pressure becomes

$$\frac{\Delta P}{P} = \frac{3(A/m)L_{\odot}a^2(1 - e^2)\sin i'}{4\pi c r_{\odot}^2 GM_{\oplus}}\left[\frac{\cos(\beta + \theta)}{1 + e\cos\theta}\right]\Bigg|_{\theta_1}^{\theta_2}. \tag{7}$$

It is convenient for numerical purposes to describe the physical characteristics of the satellite itself by a dimensionless quantity D_s such that $(A/m) = D_s\ cm^2/gm$, and also to express the ratio of perigee distance to the earth's equatorial radius by another dimensionless quantity K such that

$$q/R_{\oplus} \equiv a(1 - e)/R_{\oplus} = K \gtrsim 1.$$

Then equation (7) simplifies to

$$\frac{\Delta P}{P} = \frac{3D_s L_{\odot}R_{\oplus}^2 K^2(1 + e)\sin i'}{4\pi c r_{\odot}^2 GM_{\oplus}(1 - e)}\left[\frac{\cos(\beta + \theta)}{1 + e\cos\theta}\right]\Bigg|_{\theta_1}^{\theta_2}$$

$$= 1.40 \times 10^{-7} D_s K^2 \frac{(1 + e)}{(1 - e)}\sin i' \left[\frac{\cos(\beta + \theta)}{1 + e\cos\theta}\right]\Bigg|_{\theta_1}^{\theta_2}. \tag{8}$$

4. The Earth's Shadow

To evaluate the bracket in equation (8) we must know the values of the true anomaly, θ_1 and θ_2, at which the satellite enters and leaves the earth's shadow. To keep the problem tractable we assume, without appreciable error, that the shadow is a circular cylinder of radius $R_\oplus$, with axis of course in the anti-sun direction. The intersection of this cylinder with the orbit plane is a semi-ellipse, as shown in Figure 3. Its equation in the coordinate system already defined is

$$x^2 \cos^2 i' + y^2 = R_\oplus^2, \quad x \leqq 0 . \tag{9}$$

Transforming to polar coordinates and equating to the expression for the satellite's orbit, we find the values of θ_1 and θ_2 are the two solutions of

$$\frac{R_\oplus^2}{1 - \sin^2 i' \cos^2(\beta + \theta)} = \frac{a^2(1 - e^2)^2}{(1 + e \cos \theta)^2}, \quad \frac{\pi}{2} \leqq \beta + \theta \leqq \frac{3\pi}{2} . \tag{10}$$

If there are no solutions in the second and third quadrants the satellite is in sunshine all around the orbit; if there is one solution, $\theta_1 = \theta_2$, the satellite touches the shadow at only one place and spends none of its time in darkness. As before, we may substitute $q/R_\oplus = K \geqq 1$; then θ_1 and θ_2 are the two solutions of

$$(1 + e \cos \theta)^2 = K^2(1 + e)^2\left[1 - \sin^2 i' \cos^2(\beta + \theta)\right], \quad \frac{\pi}{2} \leqq \beta + \theta \leqq \frac{3\pi}{2} . \tag{11}$$

The specific values of θ_1 and θ_2 for a given high satellite can be found by computing i' and β for every few days and solving equation (11) by graphical or other approximate methods. The angle i', between the orbit normal and the sun, is given by

$$\cos i' = \cos i \sin \delta_\odot + \sin i \cos \delta_\odot \sin(\alpha_N - \alpha_\odot) , \tag{12}$$

where i is the inclination of the orbit plane to the equator, $\alpha_\odot$ and $\delta_\odot$ are the right ascension and declination of the sun, and α_N is the right ascension of the ascending node. The angle β can be found from

$$\cos \beta \sin i' = \sin \delta_\odot \sin \delta_Q + \cos \delta_\odot \cos \delta_Q \cos(\alpha_Q - \alpha_\odot) , \tag{13}$$

where α_Q and δ_Q are the right ascension and declination of perigee. When the angles of entry and exit, θ_1 and θ_2, have been found for various epochs they may then be used in equation (8) to compute predicted secular accelerations as a function of the time.

When we abandon consideration of a specific satellite and ask for a general solution of equation (11) to be substituted in equation (8) for $\Delta P/P$, the problem is formidably complex because the angles of entry and exit depend on four arbitrary parameters. Let us therefore first consider some specific applications which will perhaps elucidate the effect of radiation pressure on the period changes of a high satellite and then proceed to develop a quasi-general solution as a power series in the eccentricity. (For objections to this procedure, see Kozai, 1961, especially page 30.)

5. Special Cases of Orbital Orientation and Shape

(a) Orbit normal pointing toward the sun.--Here $i' = 0°$ and therefore the right side of equation (8) is zero. Physically, no satellite can spend a finite fraction of its time in the earth's shadow, even if $K = 1$.

(b) Satellite in sunshine all around the orbit.--If i' is sufficiently small the satellite will see the sun above the earth's horizon continually, as was the case with Echo I (1960 ι1) during its first two weeks aloft in August, 1960. Under such conditions, even if i' is not zero, $\Delta P/P = 0$. Mathematically, the left side of equation (6) is to be evaluated from 0 to 2π, and the result is zero. Regarded physically, the perturbing effects cancel on opposite sides of the orbit. The null result here is like that of the solar gravitational perturbation of a satellite, where there is never a "shadow" in which to hide: the secular acceleration is zero when $\vec{f}$ is regarded as a constant vector relative to the satellite orbit.

(c) The circular orbit, i' arbitrary.--To find the period change of a satellite in a circular orbit and spending time in the earth's shadow, set $e = 0$ in equation (8) and note that, by symmetry, $(\beta + \theta_2) = 2\pi - (\beta + \theta_1)$. Evaluating the bracket, we see that $\Delta P/P = 0$. Because of the symmetry the momentum loss while the satellite is moving toward the solar hemisphere is just balanced by the momentum gain when it is moving toward the opposite hemisphere.

(d) The angle $\beta = 0°$ or $180°$, i' arbitrary.--By symmetry, $\theta_2 = 2\pi - \theta_1$. The bracket in equation (8) is therefore zero and hence $\Delta P/P = 0$ for this case also, the interpretation being similar to that of case (c).

(e) The asymmetric case with $i' = 90°$, $\beta = 90°$ or $270°$.--It may be thought from all the foregoing that radiation pressure has no effect at all on satellite accelerations. The present example is intended to show otherwise and is also one that can be evaluated without great difficulty. When $\beta = 90°$ the solution of equation (11) gives $\cos\theta_1 = [K(1 + e) - e]^{-1}$ and $\cos\theta_2 = -[K(1 + e) + e]^{-1}$. Substitution in equation (8) shows that for $i' = 90°$, $\beta = 90°$, the secular acceleration is

$$\frac{\Delta P}{P} = -1.40 \times 10^{-7} D_s U(K, e) ,$$

$$U(K, e) = \frac{K\sqrt{1 + e}}{1 - e}\left\{\sqrt{(K^2 - 1) + e(K^2 + 1) + 2Ke} - \sqrt{(K^2 - 1) + e(K^2 + 1) - 2Ke}\right\}, \tag{14}$$

the period decreasing with the time. If $\beta = 270°$ the effect is equal and opposite, the period increasing secularly. Although I have not tried to prove it, this special example probably reveals about the maximum secular acceleration to be expected from radiation pressure. For one thing, with $i' = 90°$ the full force of sunlight is in the orbit plane; for another, with $\beta = 90°$ or $270°$ the effects of asymmetry are large. Table 1 presents numerical values of $U(K, e)$ for several relevant values of the eccentricity, e, and the perigee distance, $K = q/R_\oplus$.

TABLE 1

VALUES OF U(K, e) FOR USE WITH EQUATIONS (14)

K	e = 0.00	0.01	0.02	0.05	0.10	0.15	0.20	0.30	0.40	0.50
1.00	0.00	0.20	0.29	0.48	0.74	0.98	1.22	1.78	2.49	3.46
1.05	0.00	0.06	0.12	0.27	0.49	0.71	0.93	1.45	2.10	2.99
1.10	0.00	0.05	0.10	0.23	0.45	0.66	0.88	1.39	2.04	2.94
1.15	0.00	0.05	0.09	0.22	0.42	0.63	0.86	1.37	2.04	2.94
1.20	0.00	0.04	0.09	0.21	0.41	0.62	0.85	1.37	2.05	2.98
1.30	0.00	0.04	0.08	0.20	0.41	0.62	0.86	1.40	2.11	3.09
1.40	0.00	0.04	0.08	0.20	0.41	0.63	0.88	1.45	2.19	3.22
1.50	0.00	0.04	0.08	0.20	0.42	0.65	0.91	1.51	2.29	3.38

Inspection of the table reveals several points. First, the secular acceleration is zero for circular orbits of all sizes, as expected. Second, for any fixed value of the perigee distance U(K, e) increases monotonically with e, because of the increasing asymmetry of the passage through the earth's shadow. Third, for any fixed value of the eccentricity U(K, e) has a relative maximum for the smallest possible orbit (K = 1), falls to an absolute minimum at some intermediate value of K, and then rises once again. Interpreted physically, for a fixed e there is maximum asymmetry for K = 1, while there is none at all as $K \rightarrow \infty$ and the earth's shadowing effect becomes infinitesimal. With increasing K, however, the decrease in asymmetry is compensated and then overtaken by the decreasing gravitational control of the earth on the satellite. When U(K, e) is differentiated partially with respect to K, it is found that the absolute minimum occurs at a perigee of

$$K^2 = \frac{(5 + 4e^2) + 3\sqrt{1 + 8e^2}}{4(1 + e)^2} \quad . \tag{15}$$

For any eccentricity in the range $0.0 \leq e \leq 0.5$ the minimum value of U(K, e) may be found by adopting the smallest value in any given column of Table 1, because the absolute minima for all these eccentricities occur within the range of K that is tabulated. Thus for this particular orientation and independently of K, the magnitude of $\Delta P/P$ ranges from zero for a circular orbit to at least $0.5 \times 10^{-7} D_s$ at e = 0.10, at least $1.1 \times 10^{-7} D_s$ at e = 0.20, and at least $4.1 \times 10^{-7} D_s$ at e = 0.50.

(f) The orbit of low eccentricity. --A quasi-general solution for the secular acceleration may be developed as a power series in e. In equations (8) and (11) make the substitution $\Phi = \beta + \theta$, where r and Φ are polar coordinates in the orbit plane as defined by the coordinate system of section 3, such that $x = r \cos \Phi$, $y = r \sin \Phi$. Next circumscribe a circle of radius q around the center of the earth, as shown in Figure 4. In the figure, $\angle JCA = \Phi_1$ and $\angle JCB = \Phi_2$. Define $\Phi_1 = \Phi_{01} + u_1$, $\Phi_2 = \Phi_{02} + u_2$, where u_1 and u_2 are small quantities if the eccentricity is small and zero for a circular orbit. As can be seen from Figure 4, $\phi_{01} = \angle JCA_0$ and must lie in the second quadrant, while $\phi_{02} = \angle JCB_0$ and must lie in the third quadrant. The solution of equation (11) with e = 0 gives

$$\cos \phi_{01} = \cos \phi_{02} = -\frac{\sqrt{K^2 - 1}}{K \sin i'} \equiv -\nu , \quad \nu \geq 0 ,$$

$$\sin \phi_{01} = -\sin \phi_{02} = \frac{\sqrt{1 - K^2 \cos^2 i'}}{K \sin i'} \equiv \mu , \quad \mu \geq 0 . \tag{16}$$

Now set $u_1 = a_1 e + a_2 e^2 + \dots$, $u_2 = b_1 e + b_2 e^2 + \dots$, substitute each in equation (11), and then equate coefficients of like powers of e in order to obtain expressions for $a_1, a_2, \dots$, $b_1, b_2, \dots$. Then return to equation (8) for the secular acceleration and expand it similarly as a power series in e, a series which will of course contain $a_1, a_2, \dots, b_1, b_2, \dots$. Substitution of the explicit expressions for these coefficients, already found above, then gives the secular acceleration as a power series in e to any degree of accuracy. Unfortunately, of course, the greater the desirable degree of accuracy, the greater is the undesirable degree of complexity in working out the coefficients. I find the quasi-general solution, as far as order e^2, to be

$$\frac{\Delta P}{P} = -1.40 \times 10^{-7} D_s Y(K, e, \beta, i') ,$$

$$Y(K, e, \beta, i') = \frac{2K^3 e\mu \sin \beta}{\sqrt{K^2 - 1}} \left[1 + 2e(1 + \nu \cos \beta) - \frac{e(1 + \nu^3 \cos \beta)}{\mu^2 (K^2 - 1)} \right] + O(e^3) , \tag{17}$$

where μ and ν are given by equations (16).

This expression has both merits and demerits. It vanishes, as it should, for a circular orbit. For a finite eccentricity it is zero for $\beta = 0°$ or $180°$, as expected from case (c). Also, if we set $i' = 90°$ and $\beta = 90°$ in equation (17), it is found to agree with the exact equation (14) when the latter is expanded to the second order in e. The acceleration predicted by each formula for this special situation is

$$\frac{\Delta P}{P} = -1.40 \times 10^{-7} D_s \left[\frac{2K^2 e}{\sqrt{K^2 - 1}} \left\{ 1 + \frac{e(K^2 - 2)}{(K^2 - 1)} \right\} \right] . \tag{18}$$

Although equation (17) is consistent with some of the foregoing special cases, it arouses suspicion on at least two grounds. First consider a time when $\mu = 0$ and therefore by equation (16) $\phi_{01} = \phi_{02} = 180°$. The final term in equation (17) suggests an infinite secular acceleration under these conditions. Actually this particular situation occurs when the circumscribed circle of Figure 4 touches the projected shadow at one and only one point -- on the anti-sun axis. It is readily seen, therefore, that no satellite travelling on any orbit whose inscribed circle is such that $\mu = 0$ can spend a finite fraction of its period in darkness. The difficulty here is a mathematical one rather than a physical one, and as has already been pointed out $\Delta P/P = 0$ on all occasions when equation (11) has less than two solutions. A second, and legitimate, suspicion is aroused when $K = 1$ in equation (17). It is not enough to dismiss this problem with the comment that any satellite with $K = 1$ is itself in grave trouble; we deal with this problem as case (h).

(g) The orbit of low eccentricity, $i' = 90°$. --Specializing equations (16) and (17) for those times when the radiation force lies fully in the orbit plane, we find that

$$Y(K, e, \beta, 90°) = eV(K, \beta)\left[1 + e\left\{\frac{K^2 - 2}{K^2 - 1} + \frac{\cos\beta\sqrt{K^2 - 1}}{K}\right\} + \ldots\right],$$

$$V(K, \beta) = \frac{2K^2 \sin\beta}{\sqrt{K^2 - 1}} \qquad (19)$$

Table 2 gives values of $V(K, \beta)$ for several combinations of K and β and may be used to estimate quickly the leading term of equation (19). The absolute value of the second term is less than 20 percent of the leading term for all values of β at $K = 1.05$ if $e < 0.022$, at $K = 1.10$ if $e < 0.048$, at $K = 1.15$ if $e < 0.077$, at $K = 1.20$ if $e < 0.109$, at $K = 1.30$ if $e < 0.184$, at $K = 1.40$ if $e < 0.270$, and at $K = 1.50$ if $e < 0.211$. The minimum absolute contribution of the second term occurs in the neighborhood of $K = \sqrt{2}$ and then rises again with increasing perigee distance until for very large orbits 20 percent contribution occurs at $e = 0.100$. The sign of the second term is usually negative for $K < \sqrt{2}$ and is negative for all values of β provided $K < 1.27$.

TABLE 2

VALUES OF V(K, β) FOR USE WITH EQUATIONS (19)

K	β = 0°	15°	30°	45°	60°	75°	90°
1.05	0.00	1.78	3.44	4.87	5.96	6.65	6.89
1.10	0.00	1.37	2.64	3.73	4.57	5.10	5.28
1.15	0.00	1.21	2.33	3.29	4.03	4.50	4.66
1.20	0.00	1.12	2.17	3.07	3.76	4.19	4.34
1.30	0.00	1.05	2.03	2.88	3.52	3.93	4.07
1.40	0.00	1.04	2.00	2.83	3.46	3.86	4.00
1.50	0.00	1.04	2.01	2.85	3.49	3.89	4.02

From this specific example it appears likely that equations (16) and (17) constitute an adequate approximation to the practical estimate of secular accelerations due to radiation pressure, at least for a fair variety of orbits. First, as will be shown in section 6, the effect of sunlight on $\Delta P/P$ is swamped by the effect of drag if $K < 1.12$ approximately, and it is therefore unnecessary in practice to be concerned about the nature of the solution when K is very close to unity. Second, whenever $K > 1.12$ the contribution of the second term of equation (17) is moderately small if e is not too large. Presumably the higher-order terms converge rapidly. Although an extension of the power series beyond $0(e^2)$ would be useful for more eccentric orbits, it has not been attempted in the present work.

(h) The nearly circular orbit, $i' = 90°$, $K = 1$.--When perigee is at or very near the earth's surface, the approximation leading to equation (17) breaks down. Geometrically, when $K = 1$, we have $\varphi_{01} = 90°$ and $\varphi_{02} = 270°$, and even when the eccentricity is small the angles u_1 and u_2 are of rather good size, and therefore terms beyond $0(e^2)$ are needed to obtain an adequate approximation. Alternatively, it is straightforward enough in the present special case to solve equation (11) and substitute the results in equation (8) as a power series in e. The appropriate formula, to order 3/2 in e, turns out to be

$$\frac{\Delta P}{P} = -1.40 \times 10^{-7} D_s W(e, \beta) ,$$

$$W(e, \beta) = (1 + 3e/2) \sqrt{2e} \left\{ \sqrt{1 + \sin \beta} - \sqrt{1 - \sin \beta} \right\} . \tag{20}$$

This expression vanishes for $e = 0$ and also for $\beta = 0°$ or $180°$, as we are entitled to expect from previous illustrations. Table 3 contains $W(e, \beta)$ for a few selected pairs of e and β. The approximate equation (20) with $\beta = 90°$ is identical with the exact equation (14) when $K = 1$ and the function U is expanded to order 3/2 in e. When $\beta = 90°$, equation (20) is good to 1 percent if $e < 0.08$ and to 10 percent if $e < 0.28$.

TABLE 3

VALUES OF $W(e, \beta)$ FOR USE WITH EQUATIONS (20)

e	$\beta = 0°$	15°	30°	45°	60°	75°	90°
0.00	.000	.000	.000	.000	.000	.000	.000
0.01	.000	.037	.074	.110	.144	.175	.203
0.02	.000	.054	.107	.158	.206	.251	.291
0.03	.000	.067	.133	.196	.256	.312	.362
0.04	.000	.078	.155	.229	.300	.365	.424
0.05	.000	.089	.176	.260	.340	.414	.481

6. The Competition of Radiation Pressure and Atmospheric Drag

It is well known that the instantaneous tangential acceleration of a satellite moving through a stationary atmosphere is given by

$$T = -(A/m)(C_D/2)\,\rho v^2 , \tag{21}$$

where C_D is the dimensionless drag coefficient, ρ is the atmospheric density at the point in question, and v the speed of the satellite there. When the magnitude of this perturbing acceleration is compared with that due to solar radiation pressure as given by equation (1), the ratio of the two is

$$R = \frac{4\pi c r_{\odot}^2 C_D \rho v^2}{2L_{\odot}} , \tag{22}$$

a quantity that is independent of the characteristics of the satellite itself. Setting $C_D = 2$, $\rho = J \times 10^{-16}$ gm/cm^3, adopting for v the circular velocity at 800 km, and inserting the other constants, we have

$$R \cong 1.2 J . \tag{23}$$

Thus, outside the earth's shadow the two forces are of equal magnitude at a height near 800 km for a mean state of the atmosphere (Nicolet, 1960). When the sun is active and when it is close to mid-day at the perigee of a satellite, the high atmosphere is distended and the level of equal magnitudes is above 800 km.

In equation (23), J decreases more or less exponentially with height. Therefore atmospheric drag is all-important for low satellites and less so for high ones. For example Vanguard I (1958 $\beta 2$), with $D_s = 0.21$, should by equation (14) and Table 1 show a maximum secular acceleration due to radiation pressure of $\pm 0.25 \times 10^{-7}$. Referring to equation (8), the quantity sin i' passes through a maximum on the average every 45 days, which is the time taken on the average for $(\alpha_N - \alpha_{\odot})$ to regress through 180°; the semi-amplitude of this fluctuation is about 0.1×10^{-7}. A second periodic variation is that of the angle β, which passes through a complete cycle every 2.4 years. This interval is the length of the "day" at perigee, the average time taken for $(\alpha_Q - \alpha_{\odot})$ to advance by 360°. The semi-amplitude of this long-run variation is about 0.25×10^{-7}. When the anticipated effect of radiation pressure is compared with the observed accelerations of Vanguard I (Jacchia, 1959; Briggs, 1959), it appears that the period changes can confidently be attributed to drag. The effect of radiation pressure is small, and about equal to the precision with which the accelerations can be determined; it is less than the effect of drag by a factor ranging from about 5 when perigee occurs at night to about 50 when it occurs nearly under the sun. Thus for satellites as low as Vanguard I radiation effects are to be found by analyzing such elements as perigee height (Musen, Bryant, and Bailie, 1960; Musen, 1960) rather than orbital period. The latter may be employed to deduce the structure of the thermosphere.

At greater heights the situation is more delicate. During its first two weeks aloft Echo I (1960 $\iota 1$) had a perigee height near 1500 km, several scale heights above the reference level of 800 km. For example if the scale height in this layer averages about 110 km, corresponding to T = 2000°K and a composition of atomic oxygen, equation (23) then gives $R < 1.2e^{-6} < 0.01$, so that the relative magnitude of the drag force is small.

Echo I of course spends some of its time in sunshine all around the orbit. As shown in case (b) of section 5, the effect of radiation pressure on period changes is then zero within the framework of our assumptions. On such occasions the characteristics of the thermosphere can be studied without accounting for the complications of solar radiation. There is, however, the limitation that at these times $i' \cong 0°$ and we are therefore confined to examining the twilight zone of the high atmosphere. If the diurnal sunward bulge persists at these great heights, its properties can only be deduced when Echo I has the sun near its zenith in part of its orbit and thus is in darkness half a period later.

When a very high satellite periodically transits the earth's shadow, radiation pressure then has the upper hand. The preliminary mean value of $\Delta P/P$ for Echo I during the first twelve weeks of passage through the umbra was near -4×10^{-6}. The sign accords with expectation because the angle β was in the first and second quadrants during this interval, and the amount is roughly consistent with prediction. The open circles of Figure 5 give weekly predicted accelerations and the filled circles the observed accelerations. The first seven observed points are weekly means of the day-to-day values computed under the direction of Dr. Pedro Zadunaisky; the last five observed points are relatively rough graphical accelerations by the author and are preliminary only. In order to obtain the best vertical fit of predicted and observed points it is necessary to adopt $D_s \cong 270$ in equation (17), whereas a diameter of 100 feet and a mass of 60 kilograms gives $D_s \cong 120$. Apart from this, the two curves are in reasonable agreement.

The factor of 2 or 3 by which the observed period changes exceed the predicted ones could arise from several effects. First, an error in the announced value of D_s could either aggravate or reduce the discrepancy. Second, Echo I during this time may have been passing more or less centrally through a diurnal atmospheric bulge, which would reduce the discrepancy. A third possibility is non-specular reflection from the balloon surface; complete back reflection will double the magnitude of the perturbing acceleration of equation (1), whereas isotropic reflection by each surface element into its outward hemisphere will increase equation (1) by a factor of 4/3. One definite reason why the observed negative accelerations during these three months should be greater than the predicted values is the effect of terrestrial re-radiation. Reflected sunlight was acting during these weeks to amplify the negative secular acceleration. In addition, if the infra-red radiation from the surface and atmosphere is non-isotropic in the sense of being stronger on the daylit hemisphere, this component was also acting during September and October of 1960 in the same fashion. We can, of course, unscramble some of these effects more easily when Echo I has been aloft for a longer time. Perhaps it is not too much to hope that the radiation effects can be assessed well enough so that residuals may then permit deductions on the structure of the highest atmosphere even when Echo I is encountering the earth's shadow.

7. Remarks and Conclusions

Studies of the atmosphere above about 800 km are made difficult because extraneous effects rival that of drag on the period changes of satellites. The competing effect of solar radiation pressure can be evaluated and eliminated by the use of equations (16) and (17), provided the orbital eccentricity is not too large, that D_s is well known, and that it is possible to estimate a factor by which the satellite deviates from a specular reflector. The effect of terrestrial re-radiation has not been taken into account quantitatively in this study, although its role relative to direct solar pressure may be appreciable. Two further investigations are therefore suggested: (1) extension of the power series of equation (17) for more accurate assessment of the effect of direct solar radiation pressure, and (2) calculations on the influence of re-radiation from the earth on the orbit of a satellite.

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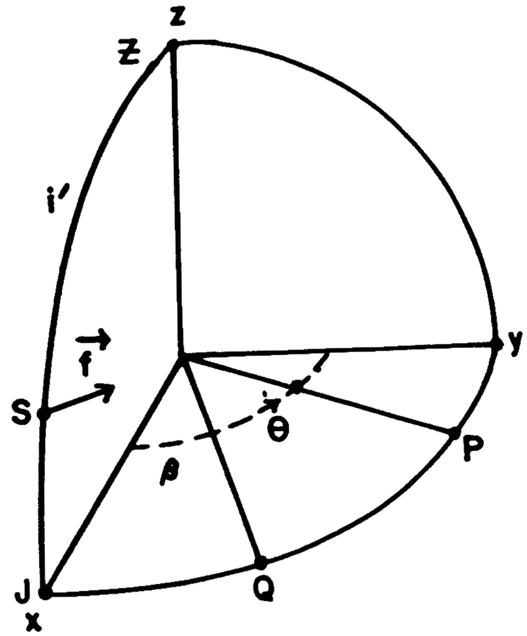

FIGURE 1. --The celestial sphere, showing the direction of the orbit normal, Z, of the sun, S, of perigee, Q, and of the instantaneous satellite position, P. The direction of the radiation vector is from S toward the origin.

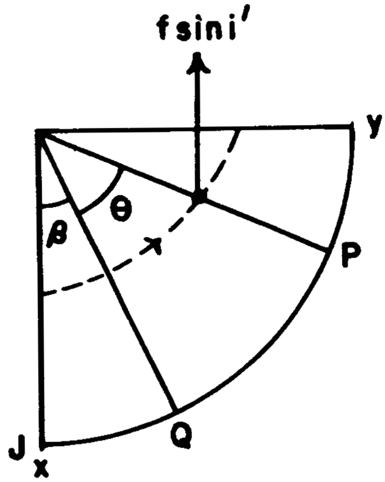

FIGURE 2. --The satellite's orbit plane, with symbols as in Figure 1. The component of the perturbing acceleration due to solar radiation pressure is f sin i' and is directed toward the negative x-axis.

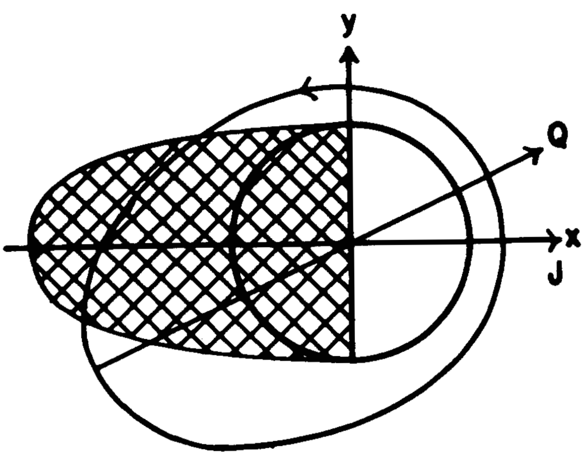

FIGURE 3. --The general geometrical relation of a satellite's orbit and the semi-ellipse of the earth's cylindrical shadow projected onto the orbit plane.

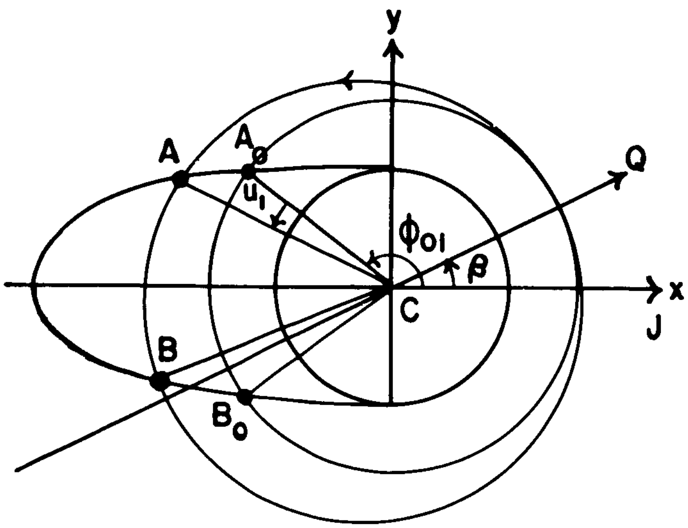

FIGURE 4. --The construction for a nearly circular orbit. The semi-ellipse of the projected shadow is shown, as are also the orbit and its inscribed circle of radius equal to the perigee distance. Notation as in section 5, case (f).

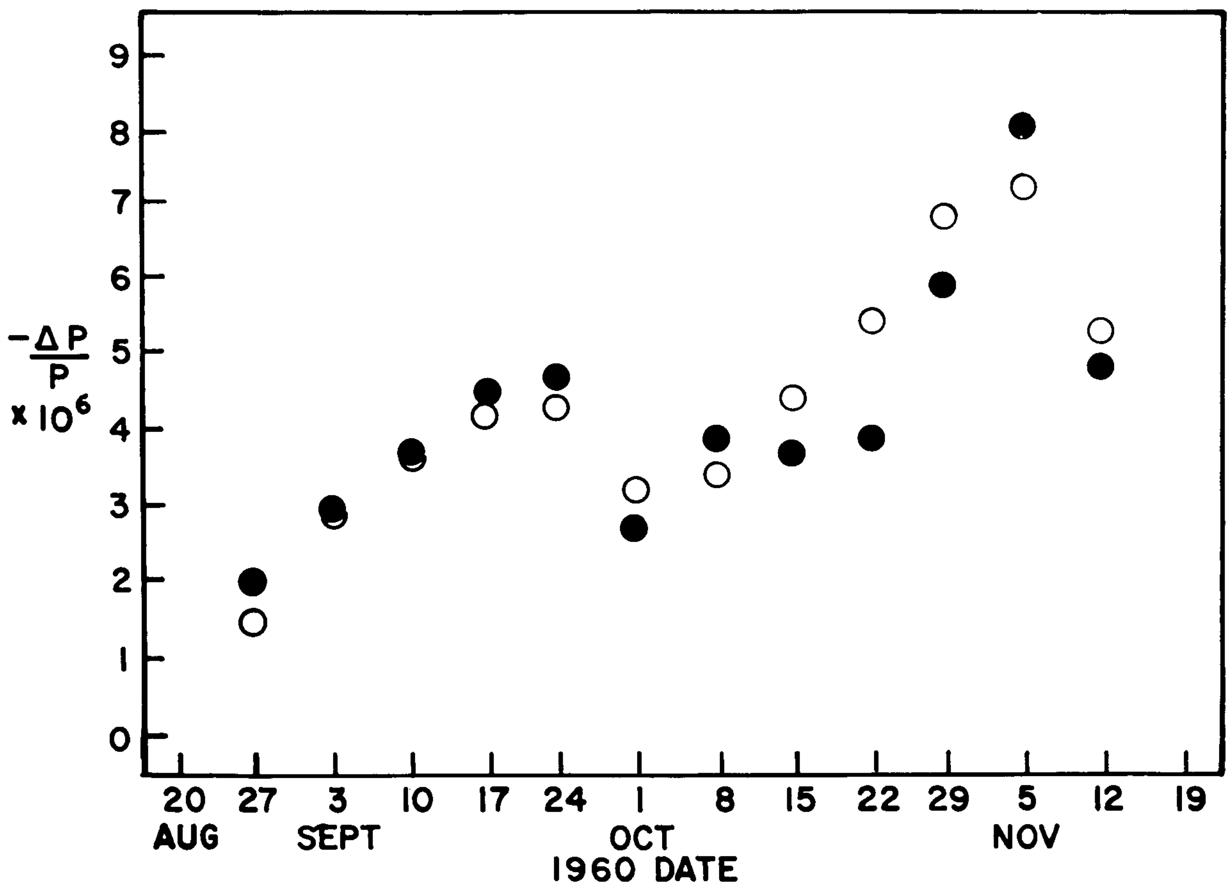

FIGURE 5. --Comparison of observation and theory. The filled circles are preliminary observed secular accelerations of Echo I during its first three months of transiting the earth's shadow; open circles are the predicted values due to solar radiation pressure.